

An $SO(10) \times S_4$ Model of Quark-Lepton Complementarity

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Abstract

The present observations of Cabibbo angle and solar mixing angle satisfy the empirical relation called Quark-Lepton Complementarity(QLC), namely $\theta_{12}^l \sim \pi/4 - \theta_C$. It suggests the existence of correlation between quarks and leptons which is supported by the idea of grand unification. We propose a specific ansatz for the structure of Yukawa matrices in $SO(10)$ unified theory which leads to such relation if neutrinos get masses through type-II seesaw mechanism. Viability of this ansatz is discussed through detailed analysis of fermion masses and mixing angles all of which can be explained in a model which uses three Higgs fields transforming as 10 and one transforming as $\overline{126}$ representations under $SO(10)$. It is shown that the proposed ansatz can be derived from an extended model based on the two pairs of 16-dimensional vector-like fermions and an S_4 flavor symmetry. The model leads to the lepton mixing matrix that is dominantly bimaximal with $\mathcal{O}(\theta_C)$ corrections related to quark mixings. A generic prediction of the model is the reactor angle $\theta_{13}^l \sim \theta_C/\sqrt{2}$ which is close to its present experimental upper bound.

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I. INTRODUCTION

Experiments on neutrino oscillations have revealed that two of the three leptonic mixing angles are large. One of them called the atmospheric mixing angle is almost maximal $\theta_{23}^l = 42.3^\circ \left({}^{+11.4}_{-7.1} \right)$ and the other known as the solar mixing angle $\theta_{12}^l = 34.5^\circ \left({}^{+3.2}_{-2.8} \right)$ is smaller compared to it [1]. In contrast, the observed quark mixing angles are small and hierarchical. The largest angle is the Cabibbo angle $\theta_C \equiv \theta_{us} \approx 13^\circ$ while other two are $\theta_{cb} \approx 2.4^\circ$ and $\theta_{ub} \approx 0.2^\circ$. An understanding of such wide dissimilarity between the quark and lepton mixing patterns is considered as one of the major challenges for the physics beyond the standard model. It has been observed long ago [2] that there exists an interesting empirical relation between quark and lepton mixing angles.

$$\theta_{12}^l + \theta_{us} \sim \frac{\pi}{4} \quad (1)$$

The above relation is known as Quark-Lepton Complementarity [3–6] and still favored by the present experimental data within their measurement errors. It is also possible to write similar relation between 23 angles of quark and lepton mixing.

$$\theta_{23}^l + \theta_{cb} \sim \frac{\pi}{4} \quad (2)$$

If such relations are not accidental, they strongly suggest the common roots between quarks and leptons [3–5]. Clearly it is very hard to realize such relations in ordinary bottom-up approaches where the quarks and leptons are treated separately with no specific connections between them. So one requires top-down approaches like the Grand Unified Theories(GUT) which sometime also unify quarks and leptons and provide a framework to construct a model in which QLC relation can be embedded in a natural way.

The general conditions under which QLC relation (1) can be realized from quark-lepton unification are thoroughly discussed in [3, 4]. We describe one such possibility here. The quark mixing matrix known as Cabibbo-Kobayashi-Maskawa(CKM) matrix is defined as $V_{CKM} = U_u^\dagger U_d$ where $U_u(U_d)$ is unitary matrix that diagonalizes the up-(down-)type quark mass matrix. Corresponding leptonic mixing matrix, also called Pontecorvo-Maki-Nakagawa-Sakata(PMNS) matrix, is $V_{PMNS} = U_e^\dagger U_\nu$. Assume that the structure of neutrino and quark mass matrices at high scale are such that the PMNS matrix is exact bimaximal $V_{PMNS} = U_{BM}$ whereas the CKM matrix is an identity matrix to a leading order.

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3)$$

Both the mixing matrices get corrected by $\mathcal{O}(\theta_C)$ terms coming from the next leading order where the down quark and charged lepton mass matrices are equal (or nearly so). In this scenario, a QLC relation can emerge from quark-lepton unification at high scale. Construction

of a realistic GUT model in which all fermion masses and mixing angles are correctly reproduced along with QLC is highly non-trivial. In fact several models [5] proposed to explain QLC are based on a smaller gauge group, namely Pati-Salam $SU(4)_c \times SU(2)_L \times SU(2)_R$ group. A complete and realistic model based on $SO(10)$ GUT has not been proposed so far. The original proposal [4] was based on $SU(5)$ relation $M_e = M_d^T$ but detailed explanation of the fermionic spectrum was not developed. Here we present a predictive $SO(10)$ based unified description of fermion masses and mixing in which QLC relation can be naturally realized.

The renormalizable models based on the $SO(10)$ gauge group are quite powerful in constraining the fermion mass structure. Moreover, they provide a natural framework for understanding neutrino masses because of the seesaw mechanisms inherent in them. Fermion masses arise in these models through their couplings to Higgs fields transforming as **10**, $\overline{\mathbf{126}}$ and **120** dimensional representations under $SO(10)$. Neutrino masses arise from two separate sources either from the vacuum expectation value (vev) of the right handed triplet (type-I) or from the left handed triplet (type-II) Higgs. The minimal model with **10** and $\overline{\mathbf{126}}$ Higgs fields has attracted a lot attention [7–9]. There also exist a class of models where appropriate flavor symmetry is integrated with $SO(10)$ framework with extended Higgs sector [10–13] to construct a predictive theory which can simultaneously explain hierarchical nature of quark masses and mixing angles and large lepton mixing angles. In this paper, we show that QLC relation follows in a specific $SO(10)$ model combined with S_4 symmetry if dominant source of neutrino mass is type-II. An additional Z_n symmetry is required in the model to get desired interactions between various fields.

The paper is organized as follows. We describe the fermion mass relations in the model based on renormalizable supersymmetric (SUSY) $SO(10)$ GUT in the next section. In section III, we propose a specific ansatz which predictively interrelates various observables of quark and lepton sectors and leads to QLC relation. We also discuss the phenomenological implications of such ansatz in this section. In section IV, we justify the proposed ansatz by a flavor symmetry group $S_4 \times Z_n$. Finally we conclude in section V.

II. RENORMALIZABLE SUSY $SO(10)$ MODEL FOR FERMION MASSES

We consider three families of **16**-dimensional fermions obtaining their masses from renormalizable couplings to four Higgs multiplets, three of them (denoted by Φ , Φ' and Φ'') transforming as **10** and the other ($\overline{\Sigma}$) as $\overline{\mathbf{126}}$ dimensional representations under $SO(10)$. The $SO(10)$ breaking can be achieved with **210** + **54** + **126** + $\overline{\mathbf{126}}$ [9]. The Yukawa interactions of the model can be written as [10]

$$W_Y = Y_{10}\psi\psi\Phi + Y_{\overline{126}}\psi\psi\overline{\Sigma} + Y_{10'}\psi\psi\Phi' + Y_{10''}\psi\psi\Phi'' \quad (4)$$

where Y_i are symmetric Yukawa coupling matrices. The representations Φ, Φ', Φ'' and $\bar{\Sigma}$ have two minimal supersymmetric standard model (MSSM) doublets in each of them. It is assumed that only one linear combination of the up-type doublets and one of the down-type doublets remain light and play the role of H_u and H_d fields. Once these light doublets acquire vacuum expectation values, they break electroweak symmetry and generate the fermion masses as well. The resulting fermion mass matrices can be suitably written as

$$\begin{aligned}
M_d &= H + F + t H' + H''; \\
M_u &= r H + s F + H' + p H''; \\
M_e &= H - 3F + t H' + H''; \\
M_D &= r H - 3s F + H' + p H''; \\
M_L &= r_L F; \\
M_R &= r_R^{-1} F.
\end{aligned} \tag{5}$$

where H, F, H' and H'' are obtained by multiplying electroweak vevs and Higgs mixing parameters with Yukawa coupling matrices $Y_{10}, Y_{\overline{126}}, Y_{10'}$ and $Y_{10''}$ respectively. r, s, t, p, r_L and r_R are dimensionless parameters determined by the Clebsch-Gordan coefficients, ratios of vevs, and mixing among Higgs fields (see [11] for example). M_D denotes neutrino Dirac mass matrix. $M_L(M_R)$ is the Majorana mass matrix for left-(right-)handed neutrinos which receives a contribution only from the vev of $\bar{\Sigma}$ field. In generic $SO(10)$ models of this type, the effective neutrino mass matrix \mathcal{M}_ν for the three light neutrinos has type-I and type-II contributions.

$$\mathcal{M}_\nu \equiv \mathcal{M}_\nu^{II} + \mathcal{M}_\nu^I = r_L F - r_R M_D F^{-1} M_D^T. \tag{6}$$

In general, both contributions are present and they depend on two different parameters so one may dominate over the other. It has been shown in several references [9] that it is possible to have symmetry breaking pattern in $SO(10)$ where type-II term dominates over the type-I contributions. In this limit, neutrino masses and mixing are governed by F which can be written as $F \sim M_d - M_e$. It is well known that this relation establish interesting relationship between $b - \tau$ unification and large atmospheric mixing angle[8]. The equations (5) and (6) are the key equations that provide basic platform to construct a model in which the QLC relation (1) can be realized.

III. ANSATZ

We propose following ansatz which leads to relation (1).

$$H = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & h & h \\ 0 & h & h \end{pmatrix}; F = \begin{pmatrix} b+c & \sqrt{2}a & 0 \\ \sqrt{2}a & b+c & 0 \\ 0 & 0 & b-c \end{pmatrix}; H' = \begin{pmatrix} 0 & 0 & \sqrt{2}a' \\ 0 & 0 & 0 \\ \sqrt{2}a' & 0 & 0 \end{pmatrix}; H'' = x I \tag{7}$$

where I is 3×3 identity matrix. To do the simple analytical study of such ansatz we assume that all the above parameters are real. Without loss of generality, we can express the above matrices in a basis with diagonal H . Such basis are obtained by rotating the **16**-dimensional fermion fields in 2-3 plane by an angle $\pi/4$. The matrices in (7) will be redefined in new basis as

$$(H, F, H', H'') \rightarrow R_{23} \left(\frac{\pi}{4} \right) (H, F, H', H'') R_{23}^T \left(\frac{\pi}{4} \right) \quad (8)$$

and can be rewritten as

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h \end{pmatrix}; F = \begin{pmatrix} b+c & a & a \\ a & b & c \\ a & c & b \end{pmatrix}; H' = \begin{pmatrix} 0 & -a' & a' \\ -a' & 0 & 0 \\ a' & 0 & 0 \end{pmatrix}; H'' = x I \quad (9)$$

Before we present the detailed analysis let us look at some immediate implications of the above ansatz. The dominant **10**-Higgs coupling matrix H has rank-1. As it was pointed out in [12, 13] this can simultaneously explain both the observed hierarchy of quark masses as well as the origin of large lepton mixings if the light neutrino masses are generated through type-II seesaw mechanism. Assuming only one **10**-Higgs H contribution in charged fermion mass matrices, we get at zeroth order,

$$m_b = m_\tau = \frac{1}{r} m_t; V_{CKM} = I; V_{PMNS} = U_{BM}. \quad (10)$$

Correct $b-\tau$ unification and large lepton mixings (bimaximal) are obtained with no mixings between quarks. The charged fermions of first two generations are massless in this case. Further, the contributions coming from other Higgs coupling matrices F, H' and H'' make the model realistic by giving nonzero masses to first two fermion generations as well as by perturbing both the mixing matrices which reproduce observed mixing patterns for both the quark and lepton sectors.

We now present the detailed analysis of ansatz(9). Substituting it in eq.(5) and (6), we get

$$M_u = \begin{pmatrix} s(b+c) + x' & sa - a' & sa + a' \\ sa - a' & sb + x' & sc \\ sa + a' & sc & rh + sb + x' \end{pmatrix}; M_d = \begin{pmatrix} b+c+x & a - ta' & a + ta' \\ a - ta' & b+x & c \\ a + ta' & c & h+b+x \end{pmatrix};$$

$$M_e = \begin{pmatrix} -3(b+c) + x & -3a - ta' & -3a + ta' \\ -3a - ta' & -3b+x & -3c \\ -3a + ta' & -3c & h - 3b + x \end{pmatrix}; M_\nu = r_L F \quad (11)$$

where $x' = px$. Since each mass matrix is real symmetric, it can be diagonalized by a rotation matrix parameterized (in the standard parameterization) by three angles.

$$R_f^T M_f R_f = \text{Diag}(m_{f1}, m_{f2}, m_{f3});$$

$$R_f = R_{23}(\theta_{23}^f) R_{13}(\theta_{13}^f) R_{12}(\theta_{12}^f) \quad (12)$$

where $f = d, u, e, \nu$ and R_{ij} is a rotation matrix in ij plane. The charged fermion mass matrices are hierarchical ($h \gg b, c \gg a, a' \gg x, x'$) and can be approximately diagonalized by Jacobi rotation. The results obtained from such diagonalization for the quark sector are displayed below.

$$\begin{aligned} m_b &\approx h + b + x + \mathcal{O}\left(\frac{c^2}{h}\right); \\ m_s &\approx b + x + \frac{(a - ta')^2}{b} \left(1 - \frac{x}{b}\right) + \mathcal{O}\left(\frac{c^2}{h}\right); \\ m_d &\approx b + c + x - \frac{(a - ta')^2}{b} \left(1 - \frac{x}{b}\right) + \mathcal{O}\left(\frac{a^2}{h}\right). \end{aligned} \quad (13)$$

$$\begin{aligned} m_t &\approx rh + sb + x' + \mathcal{O}\left(\frac{s^2 c^2}{rh}\right); \\ m_c &\approx sb + x' + \frac{(sa - a')^2}{sb} \left(1 - \frac{x'}{sb}\right) + \mathcal{O}\left(\frac{s^2 c^2}{rh}\right); \\ m_u &\approx s(b + c) + x' - \frac{(sa - a')^2}{sb} \left(1 - \frac{x'}{sb}\right) + \mathcal{O}\left(\frac{s^2 a^2}{rh}\right). \end{aligned} \quad (14)$$

$$\theta_{12}^d \approx -\frac{a - ta'}{b} \left(2 + \frac{c - x}{b}\right); \quad \theta_{23}^d \approx -\frac{c}{h}; \quad \theta_{13}^d \approx -\frac{a + ta'}{h} \left(1 + \frac{c}{h}\right). \quad (15)$$

$$\theta_{12}^u \approx -\frac{sa - a'}{sb} \left(2 + \frac{sc - x'}{sb}\right); \quad \theta_{23}^u \approx -\frac{sc}{rh}; \quad \theta_{13}^u \approx -\frac{sa + a'}{rh} \left(1 + \frac{sc}{rh}\right). \quad (16)$$

Let us underline some important points in connection with above relations.

- The six real parameters h, b, x, r, s, x' can be approximated from the six quark masses. m_b and m_s determine the parameters h and b . It is easy to see that $r \approx m_t/m_b$ and $s \approx m_c/m_s$ are required to obtain the masses of heavy quarks m_t and m_c . Further, m_d and m_u fix the values of x and x' . Since $b, c \gg x$, we require $c \sim -b$ to obtain small masses of first generation fermions.
- Let us assume that $a' \approx sa$ in order to keep $\theta_{12}^u \ll \theta_{12}^d$. Also note that $\theta_{23}^u \approx (s/r)\theta_{23}^d \ll \theta_{23}^d$ and $\theta_{13}^u \sim (s/r)\theta_{13}^d \ll \theta_{13}^d$. In this limit, the quark mixing matrix takes the form

$$V_{CKM} = U_u^\dagger U_d \approx U_d \approx R_{23}(\theta_{23}^d) R_{13}(\theta_{13}^d) R_{12}(\theta_{12}^d) \quad (17)$$

- The elements of the CKM matrix fix some more parameters as follows.

$$c \sim -V_{cb} h; \quad a - ta' \sim -V_{us} b; \quad a + ta' \sim -V_{ub} h. \quad (18)$$

An interesting relationship between V_{us} and V_{ub} can be found in the limit $t \sim 0$.

$$V_{ub} \approx V_{us} \frac{m_s}{m_b} + \mathcal{O}\left(\frac{m_s^2}{m_b^2}\right) \quad (19)$$

We will show later in this section that $t \sim 0$ is a necessary requirement to obtain QLC relation(1).

- Our assumption of real parameters makes the theory CP invariant. The observed CP violation in the quark sector can be accommodated by making some parameters complex.

It is interesting to note that all the parameters are fixed in terms of the observables of the quark sector. Hence the entire lepton sector emerges as the prediction of the model. Let us first derive the predictions for the charged leptons.

$$\begin{aligned} m_\tau &\approx h - 3b + x + \mathcal{O}\left(\frac{c^2}{h}\right); \\ m_\mu &\approx -3b + x - \frac{(3a + ta')^2}{3b} \left(1 + \frac{x}{3b}\right) + \mathcal{O}\left(\frac{c^2}{h}\right); \\ m_e &\approx -3(b + c) + x + \frac{(3a + ta')^2}{3b} \left(1 + \frac{x}{3b}\right) + \mathcal{O}\left(\frac{a^2}{h}\right). \end{aligned} \quad (20)$$

$$\theta_{12}^e \approx -\frac{3a + ta'}{3b} \left(2 + \frac{3c + x}{3b}\right); \quad \theta_{23}^e \approx \frac{3c}{h}; \quad \theta_{13}^e \approx \frac{3a - ta'}{h} \left(1 + \frac{c}{h}\right). \quad (21)$$

Noteworthy features of the above relations are the following,

- It predicts $m_\tau \approx m_b$ and $m_\mu \approx -3m_s$.
- For $b = -c$, $m_e \approx m_d$ which is viable with observed values of m_e and m_d extrapolated at the GUT scale within 3σ deviations [15]. However for $b \neq -c$, any desired value of m_d/m_e can be obtained.
- For $t \sim 0$, $\theta_{12}^e \approx \theta_C$, $\theta_{23}^e \approx -3\theta_{cb}$ and $\theta_{13}^e \approx -3\theta_{ub}$.

The light neutrino mass matrix in eq.(11) has the most general form which can be diagonalized by bimaximal matrix U_{BM} . The mass eigenvalues are,

$$m_1 = m_0(b + c + \sqrt{2}a); \quad m_2 = m_0(b + c - \sqrt{2}a); \quad m_3 = m_0(b - c) \quad (22)$$

Interestingly, for $b = -c$ (which can now also be written as $V_{cb} \approx m_s/m_b$), we get the partial degenerate neutrino mass spectrum $m_1 = -m_2 \ll m_3$ which leads to vanishing solar (mass)² difference ($\Delta m_{sol}^2 = m_2^2 - m_1^2 = 0$) at high scale. We performed numerical study and found

that the radiative corrections to the original neutrino mass matrix are unable to generate the required splitting between m_1 and m_2 . Another way to induce non zero value of Δm_{sol}^2 is to allow type-I contribution to the original type-II seesaw neutrino mass matrix. However such contribution is highly hierarchical (like M_t^2) and it largely contributes to the 33 element of neutrino mass matrix which ultimately spoils the nice symmetry of neutrino mass matrix and hence the bimaximality of neutrino mixings. This forces us to consider the case where $V_{cb} \neq m_s/m_b$. In this case we obtain the following expression for the ratio of the solar to atmospheric squared mass difference.

$$\frac{\Delta m_{sol}^2}{\Delta m_{atm}^2} \approx \sqrt{2} V_{us} \left(\frac{m_s/m_b}{V_{cb}} \left(1 + \frac{m_s}{m_b} \right) - 1 \right) \quad (23)$$

Note that one requires $m_s/m_b \sim 1.08 V_{cb}$ to obtain the observed value of $\Delta m_{sol}^2/\Delta m_{atm}^2$ (~ 0.031) and it implies that $V_{cb} < m_s/m_b$ which is not favored by their present observed values extrapolated at the GUT scale. However as argued in [13], the threshold corrections to $b-s$ quark mass mixing from gluino and wino exchange via one-loop diagrams can give desired value of V_{cb} . The required deviation from $b = -c$ is quantified by

$$b + c \approx m_s \left(1 - \frac{V_{cb}}{m_s/m_b} \right) \lesssim 0.08 m_s$$

which is small and of order of first generation fermion masses and hence allows the correct m_d in eq.(13).

The leptonic mixing matrix can be seen as dominant bimaximal mixing resulting from neutrino mass matrix and then corrected by $\mathcal{O}(\theta_C)$ terms coming from the unitary matrix U_e which diagonalize charged lepton mass matrix.

$$V_{PMNS} \equiv U_e^\dagger U_\nu = U_e^T U_{BM} \quad (24)$$

where $U_e = R_{23}(-3\theta_{cb})R_{13}(-3\theta_{ub})R_{12}(\theta_C)$. The resulting neutrino mixing parameters are the following.

$$\begin{aligned} U_{e2} &\equiv (V_{PMNS})_{12} \approx -\frac{1}{\sqrt{2}} + \frac{(V_{us} - 3V_{ub})}{2}; \\ U_{\mu 3} &\equiv (V_{PMNS})_{23} \approx -\frac{1}{\sqrt{2}}(1 + 3V_{cb}); \\ U_{e3} &\equiv (V_{PMNS})_{13} \approx -\frac{1}{\sqrt{2}}(V_{us} + 3V_{ub}). \end{aligned} \quad (25)$$

The correction of $\mathcal{O}(\theta_C)$ from charged lepton generates correct solar mixing angle which follows QLC relation (1). The atmospheric mixing angle gets considerable deviation $\theta_{23}^l \approx \frac{\pi}{4} + 3\theta_{cb}$ in this model unlike the standard QLC relation for 23 mixing angle of quark and lepton given in eq.(2). The model also predicts large value of $U_{e3} \approx 0.16$ which

can be tested in planned long baseline experiments.

Note that eq.(25) holds at GUT scale which might be changed by RGE corrections in principle. However it is known that running of the Cabibbo angle is negligibly small in MSSM even with large value of $\tan\beta$. Running of leptonic mixing angle depends on the type of mass spectrum of light neutrinos. For $b \neq -c$, neutrino mass spectrum follows normal hierarchy $m_1 < m_2 \ll m_3$. The effect of RGE corrections are known to be negligible in this case and eq.(25) holds at low scale also.

We now provide an example of values of the parameters of eq.(9) which successfully generate entire fermion mass spectrum as well as mixing patterns for both quark and lepton sector. The required CP violation in the quark sector is incorporated by making a' complex. In the limit $t \sim 0$, a' contributes only to the up quark mass matrix and does not change the other predictions of ansatz given in eq.(9). One more parameter x' is made complex to reproduce m_u correctly. The numerical values of parameters are

$$\begin{aligned} h &= 1.7 \text{ GeV}; \quad b = 0.0243 \text{ GeV}; \quad c = -0.022113 \text{ GeV}; \quad a = -0.0052 \text{ GeV}; \\ a' &= (0.0344247 - 0.028885i) \text{ GeV}; \quad x' = (0.0233596 - 0.00293374i) \text{ GeV}; \\ x &= 0.00325 \text{ GeV}; \quad r = 55.88; \quad s = -8.64198; \quad t = 0. \end{aligned} \quad (26)$$

Substituting these numbers in eq.(11), we get

$$\begin{aligned} m_t &= 94.8 \text{ GeV}; \quad m_c = 0.19 \text{ GeV}; \quad m_u = 0.65 \text{ MeV}; \\ m_b &= 1.73 \text{ GeV}; \quad m_s = 28.5 \text{ MeV}; \quad m_d = 4.21 \text{ MeV}; \\ m_\tau &= 1.63 \text{ GeV}; \quad m_\mu = 75.4 \text{ MeV}; \quad m_e = 0.35 \text{ MeV}. \end{aligned} \quad (27)$$

$$\begin{aligned} \sin\theta_{us} &= 0.222; \quad \sin\theta_{cb} = 0.015; \quad \sin\theta_{ub} = 0.005; \quad \delta_{CKM} = 60.9^\circ; \\ \sin^2\theta_{12}^l &= 0.368; \quad \sin^2\theta_{23}^l = 0.527; \quad \sin^2\theta_{13}^l = 0.024; \quad \frac{\Delta m_{sol}^2}{\Delta m_{atm}^2} = 0.030. \end{aligned} \quad (28)$$

The obtained spectrum is in good agreement with the data extrapolated at the GUT scale. For example, we compare our results with the charged fermion masses obtained at the GUT scale in the MSSM for $\tan\beta=55$, $M_{SUSY} = 1 \text{ TeV}$ and $M_{GUT} = 2 \times 10^{16} \text{ GeV}$ given in table 5 of reference [15]. All charged fermion masses (except m_d) obtained here fits with the data within 1σ . Our ansatz predicts larger value of m_d . The quark mixing angles θ_{cb} is small ($< m_s/m_b$) as required by eq.(23). The reproduced values of lepton mixing angles and $\Delta m_{sol}^2/\Delta m_{atm}^2$ are also in accordance with their updated low energy values (within 3σ measurement errors) given in [1].

IV. THE MODEL

In this section, we will illustrate how the ansatz (7) can be obtained in a model from flavor symmetry. We use discrete flavor symmetry based on the group S_4 which is a group

of permutation of four distinct objects. It has 24 distinct elements filled in five conjugate classes and hence five irreducible representations of dimensions $\mathbf{3_2}, \mathbf{3_1}, \mathbf{2}, \mathbf{1_2}$ and $\mathbf{1_1}$. A singlet representation with subscript “2” changes sign under transformation involving the odd number of permutations of S_4 . More details on the group theory of S_4 , its multiplication rules and the Clebsch-Gordan coefficients are reported in [14].

Our model follows the same line as model constructed in [13] and uses the same symmetry group. However it differs at some places since the ansatz required here is different from their ansatz. The basic matter fields and Higgs fields content of the model is the same as discussed in section II. In addition to this we use five flavon fields which are singlets under $SO(10)$ and two pair of vector-like fermion fields which transform like $\mathbf{16} \oplus \overline{\mathbf{16}}$ under $SO(10)$. We impose the S_4 symmetry together with Z_n symmetry to get desired structure of Yukawa matrices. Three matter fields ψ are assigned as $\mathbf{3_2}$ dimensional representation of S_4 while five flavon fields χ, ϕ, η, σ and σ' form $\mathbf{3_1}, \mathbf{3_2}, \mathbf{3_1}, \mathbf{1_1}$ and $\mathbf{1_2}$ representations of S_4 respectively. The other fields are singlet ($\mathbf{1_1}$ or $\mathbf{1_2}$) under S_4 . An additional Z_n symmetry is required to allow/forbid interactions between particular fields. The Z_n charges of various fields are listed in table(I) where $\omega = e^{i(2\pi/n)}$.

	ψ	Φ	Φ'	Φ''	$\overline{\Sigma}$	χ	ϕ	η	σ	σ'	Ψ_{V1}	$\overline{\Psi}_{V1}$	Ψ_{V2}	$\overline{\Psi}_{V2}$
$SO(10)$	$\mathbf{16}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{10}$	$\overline{\mathbf{126}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{16}$	$\overline{\mathbf{16}}$	$\mathbf{16}$	$\overline{\mathbf{16}}$
S_4	$\mathbf{3_2}$	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{1_1}$	$\mathbf{1_1}$	$\mathbf{3_1}$	$\mathbf{3_2}$	$\mathbf{3_1}$	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{1_1}$	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{1_2}$
Z_n	1	ω^{-2m}	$\omega^{-(p+q)}$	ω^{-2q}	ω^{-2k}	ω^k	ω^m	ω^p	ω^k	ω^q	ω^m	ω^{-m}	ω^k	ω^{-k}

TABLE I. Various fields and their representations under $SO(10) \times S_4 \times Z_n$.

Let us consider a theory above GUT scale which is invariant under the symmetry group $SO(10) \times S_4 \times Z_n$. The Yukawa superpotential allowed by such symmetry can be written as

$$\begin{aligned}
W = & (\phi\psi)\overline{\Psi}_{V1} + \lambda\Psi_{V1}\Psi_{V1}\Phi + M_1\Psi_{V1}\overline{\Psi}_{V1} \\
& + (\chi\psi)\overline{\Psi}_{V2} + \lambda'\Psi_{V2}\Psi_{V2}\overline{\Sigma} + M_2\Psi_{V2}\overline{\Psi}_{V2} \\
& + \sum_i \frac{\alpha_i}{\Lambda^2}(\chi^2\psi\psi)_i\overline{\Sigma} + \frac{\beta}{\Lambda^2}\sigma(\chi\psi\psi)\overline{\Sigma} + \frac{\gamma}{\Lambda^2}\sigma^2(\psi\psi)\overline{\Sigma} \\
& + \frac{\alpha'}{\Lambda^2}\sigma'(\eta\psi\psi)\Phi' + \frac{\alpha''}{\Lambda^2}\sigma'^2(\psi\psi)\Phi''
\end{aligned} \tag{29}$$

where Λ is the Planck scale up to which the theory is valid. The S_4 singlet contraction of flavor index is indicated with bracket. $\alpha_i, \alpha', \alpha'', \beta, \gamma, \lambda$, and λ' are coefficients of $\mathcal{O}(1)$. The term $(\chi^2\psi\psi)_i$ represents all the different S_4 contractions which can be constructed as follows:

$$\begin{aligned}
(\chi^2\psi\psi)_i \equiv & ((\chi\chi)_{\mathbf{1_1}}(\psi\psi)_{\mathbf{1_1}}), ((\chi\chi)_{\mathbf{2}}(\psi\psi)_{\mathbf{2}}), ((\chi\chi)_{\mathbf{3_1}}(\psi\psi)_{\mathbf{3_1}}), ((\chi\chi)_{\mathbf{3_2}}(\psi\psi)_{\mathbf{3_2}}), \\
& ((\chi\psi)_{\mathbf{1_2}}(\chi\psi)_{\mathbf{1_2}}), ((\chi\psi)_{\mathbf{2}}(\chi\psi)_{\mathbf{2}}), ((\chi\psi)_{\mathbf{3_1}}(\chi\psi)_{\mathbf{3_1}}), ((\chi\psi)_{\mathbf{3_2}}(\chi\psi)_{\mathbf{3_2}})
\end{aligned} \tag{30}$$

where $(\dots)_R$ indicates the representation R under S_4 . Now consider a theory below the scale of $M_{1,2}$ and at the GUT scale. The effective superpotential after integrating out heavy vector-like fields is given by,

$$\begin{aligned}
W_{eff} = & \frac{\lambda}{M_1^2}(\phi\psi)(\phi\psi)\Phi + \frac{\lambda'}{M_2^2}(\chi\psi)(\chi\psi)\bar{\Sigma} \\
& + \sum_i \frac{\alpha_i}{\Lambda^2}(\chi^2\psi\psi)_i\bar{\Sigma} + \frac{\beta}{\Lambda^2}\sigma(\chi\psi\psi)\bar{\Sigma} + \frac{\gamma}{\Lambda^2}\sigma^2(\psi\psi)\bar{\Sigma} \\
& + \frac{\alpha'}{\Lambda^2}\sigma'(\eta\psi\psi)\Phi' + \frac{\alpha''}{\Lambda^2}\sigma'^2(\psi\psi)\Phi''
\end{aligned} \tag{31}$$

where first two terms allow the desired rank-1 structure of Yukawa matrices. Note that effective Yukawa superpotential still has the symmetry $SO(10) \times S_4 \times Z_n$. This symmetry will be broken to $SO(10)$ by vevs of the flavon fields. In order to get the desired structure of Yukawa couplings, we will choose particular vacuum alignment of the flavon fields as given below.

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} v_\phi; \quad \langle\chi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_\chi; \quad \langle\eta\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_\eta; \quad \langle\sigma\rangle = v_\sigma; \quad \langle\sigma'\rangle = v_{\sigma'} \tag{32}$$

These vevs of flavon fields break flavor symmetry S_4 at the GUT scale and generate following structure of various Yukawa couplings.

$$Y_{10} = \frac{\lambda v_\phi^2}{M_1^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{33}$$

$$Y_{126} = \frac{\lambda' v_\chi^2}{M_2^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{v_\chi^2}{\Lambda^2} \begin{pmatrix} \tilde{\alpha} & 0 & 0 \\ 0 & \tilde{\alpha} & 0 \\ 0 & 0 & \tilde{\alpha}_0 \end{pmatrix} + \frac{\beta v_\chi v_\sigma}{\Lambda^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\gamma v_\sigma^2}{\Lambda^2} I \tag{34}$$

$$Y_{10'} = \frac{\alpha' v_{\sigma'} v_\eta}{\Lambda^2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{35}$$

$$Y_{10''} = \frac{\alpha'' v_{\sigma'}^2}{\Lambda^2} I \tag{36}$$

where all non-relevant Clebsch-Gordan coefficients are suitably absorbed. $\tilde{\alpha}$ and $\tilde{\alpha}_0$ are linear combinations of different α_i . The Yukawa matrices derived from the super potential can successfully explain the ansatz given in eq.(7). Note that $M_2 \ll \Lambda$ which implies $b + c \ll b - c$ (or $b \approx -c$) in eq.(7). Further, the assumption $M_1 \ll M_2$ leads to $h \gg b, c$.

It is very important to show that the required vacuum structure of flavon fields (32) is allowed by flavon superpotential. This point has already been discussed in great details in reference [13]. Since our model has the same kind of flavon structure as theirs, we simply use their results. Note that due to non-trivial Z_n charges, bilinear terms which correspond to masses of flavon fields are not allowed. As a result of this the model requires doubling of flavon fields to allow Dirac type mass terms. The new flavon fields have the same S_4 representations but opposite Z_n charges. It has been shown in [13] that all the desired vacua of eq.(32) are present in the model.

V. SUMMARY

In this paper, we have studied a possible way to realize QLC relation (1) between the Cabibbo angle and solar mixing angle in realistic quark-lepton unification theory based on $SO(10)$ gauge group. We have shown here that it is indeed possible to obtain such relation starting from the fermionic mass structure (5) if they are supplemented with ansatz (7) and assuming that only type-II seesaw mechanism is responsible for light neutrino masses. One necessary ingredient for QLC is bimaximal mixing pattern from the neutrino sector which has been obtained through specific ansatz. Our ansatz also makes use of recently proposed [12] rank-1 strategy which naturally explains charged fermions mass hierarchy as well as origin of hierarchical quark mixing angles as opposed to the large lepton mixing angles. We have shown through the detailed analysis that this ansatz is capable of explaining the entire fermionic spectrum and not just the QLC relation. Moreover, the various predictions made by such ansatz are in agreement with observations. We have shown that the proposed ansatz can be obtained in a model from a discrete flavor symmetry group S_4 together with an additional Z_n symmetry. A generic prediction of our approach is $\theta_{13}^l \approx \theta_C/\sqrt{2}$ which is near to its current experimental upper bound. The atmospheric mixing angle gets considerable deviation from maximality ($\theta_{23}^l \approx \pi/4 + 3\theta_{cb}$) in this approach. These predictions can be confirmed or excluded by the current generation of neutrino oscillations experiments.

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